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Existence and Stability of Steady Fronts in Bistable Coupled Map Lattices

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We prove the existence and we study the stability of the kinklike fixed points in a simple coupled map lattice (CML) for which the local dynamics has two stable fixed points. The condition for the existence allows us to define a critical value of the coupling parameter where a (multi) generalized saddle-node bifurcation occurs and destroys these solutions. An extension of the results to other CMLs in the same class is also displayed. Finally, we emphasize the property of spatial chaos for small coupling.

KEY WORDS: Coupled map lattices; front; saddle-node bifurcation.

The dynamics of localized structures is known to be one of the most relevant features of the extended dynamical systems. These particular solutions usually manifest themselves as solitons, interfaces, fronts, kinks, or domain walls separating two regions of the space where the time evolution is homogeneous (or at least regular), namely the domains. The kink propagation in space is often invoked as a destabilization factor for the stable domains and is therefore designed to be the origin of disorder in the underlying system. With the dynamics of spatial wavelenghts, the kink dynamics are thus the main components to be analyzed in the framework of spatiotemporal intermittency.[1,2]

Various models for the dynamics of large systems have been proposed.[3] Most of them consist of partial differential equations (PDEs) such as the Ginzburg–Landau or the Swift–Hohenberg equation. In this

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framework the problem of the dynamics of fronts is now well-understood in the cases where one domain is stable whereas the other is unstable.[4,5] Furthermore, models of ordinary differential equations (ODEs) coupled via a discrete Laplacian were also introduced as supports for discrete space dynamics (e.g., the nonlinear Schrödinger equation). With these systems, the dynamics of interfaces was analytically investigated as a space discretization problem, and hence with the assumption of being governed by PDEs, and the solutions exhibit good agreement with the experimental and numerical data.[6,7]

In this article, we propose an alternate description of the kink dynamics in a (one-dimensional) space-time discrete dynamical system with a continuous state, namely the coupled map lattice (CML).[8,9] The simplest kinks, that is, the fronts in bistable systems, are studied. The numerical simulations reveal the so-called "propagation failure". For a nonsymmetric local dynamics, the fronts are stationary solutions until a particular value of the coupling strength is reached.[10] Above this value, the fronts propagate in the lattice with a traveling wave-like behavior. The same behavior is observed in the front (between two stable domains) solutions of PDEs. However, the interfaces in these models are moving for all the values of the coupling strength. The difference between both models, the effect of pinning, can then be assigned to a space discreteness. This effect is also reminiscent of various phenomena in condensed matter physics such as the Peierls–Nabarro barrier in the Frenkel–Kontorova model of dislocations.[11]

The main goal of this paper is to prove the existence of the steady front solutions in a simple CML, until a particular value of the diffusion coefficient is reached, where a (multi) generalized saddle-node bifurcation occurs. In condensed matter physics, the effect of pinning is explained using a two dimensional area-preserving map which represents the action of a dynamical system in space. Following the same idea, we show how it is possible to construct explicitly the kink solution. The properties of the computed solutions are then examined in order to describe the instability which is at the origin of the front propagation. These results are extended to another local map and a numerical investigation is proposed in the case of a continuous nonlinear map. Finally, we focus our attention on the other types of fixed points in these systems.

1. DEFINITIONS

The "physical space" of the CML under consideration is chosen to be the infinite one-dimensional lattice $\mathbb{Z}$. The phase space is the direct product $M=[0,1]^{\mathbb{Z}}$. A point $x \in M$ is written $x=(x_i)_{i\in\mathbb{Z}}$. We will give below a

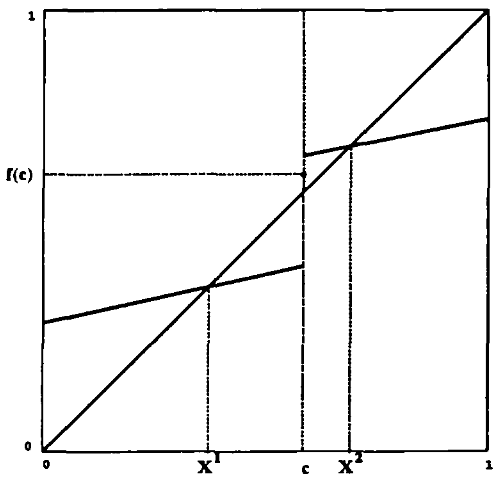

Fig. 1. The local map f. The plot is for the situation where $c > (X^1 + X^2)/2$ and $\varepsilon < \varepsilon_\beta$ (see Section 4).

norm on this phase space in order to give it the structure of a closed subspace of a Banach space.

The CML is a one-parameter family of mappings:

$$F_\varepsilon\colon\quad M \to M$$
$$x^t \mapsto x^{t+1}$$

where x^t denotes the state of the system at time t. The model is to be representative of the simplest reaction–diffusion systems, that is, when the spatial interaction is just the usual (discrete) Laplacian operator. The new state at time $t+1$ is then given by the following convex linear combination:

$$x_i^{t+1} = (F_\varepsilon x^t)_i = (1-\varepsilon)\, f(x_i^t) + \frac{\varepsilon}{2}\,(f(x_{i-1}^t) + f(x_{i+1}^t)) \qquad \forall i \in \mathbb{Z}$$

The parameter $\varepsilon \in [0, 1]$ is the diffusion coefficient.

The (nonlinear) local map is taken to be the simplest map of the interval that is bistable. To be precise, we introduce a multiparameter family of piecewise linear mappings (Fig. 1):

$$f_{\{\mu\}}\colon\quad [0, 1] \to [0, 1]$$
$$x \mapsto f(x)$$

such that $\{\mu\} = \{a, \alpha, \beta, c, \varepsilon\}$ and

$$f(x) = \begin{cases} ax + \alpha & \text{if } 0 \leqslant x < c \\ f(c) & \text{if } x = c \\ ax + \beta & \text{if } c < x \leqslant 1 \end{cases} \tag{1}$$

where

$$f(c) = \min \left\{ \frac{2ac + (\alpha + \beta)\{1 - [1 + 2a\varepsilon/(1-a)]^{1/2}\}}{2\{1 - [(1-a)(1-a+2a\varepsilon)]^{1/2}\}}, ac + \beta \right\}$$

depends on ε. The $f(c)$ is defined in such a way that, as will be shown later, there always exists an unstable kinklike fixed point when the corresponding stable one exists. The parameters a, α, β and c obey the following inequalities:

$$\begin{cases} 0 < a < 1 \\ 0 < \alpha < \beta < 1 \\ \alpha < c(1-a) < \beta \end{cases}$$

These ensure the existence of two stable fixed points for f:

$$X^1 = \frac{\alpha}{1-a} \qquad \text{and} \qquad X^2 = \frac{\beta}{1-a}$$

For the sake of simplicity, we define f so that these fixed points are the only attractors. The choice of this particular map is motivated by its simplicity which allows us to handle analytically some aspects of the CML dynamics. Notice that we are not dealing with a simpler CML where the local map is piecewise constant, that is, $a = 0$, because in that case, the model reduces to a cellular automata model, that is, to a finite set of states.

It is also possible to compute the kinklike fixed points for a more general situation where the local dynamics is continuous:

$$g(x) = \begin{cases} ax + \alpha & \text{if } 0 \leqslant x < c_1 \\ cx + \gamma & \text{if } c_1 \leqslant x \leqslant c_2 \\ ax + \beta & \text{if } c_2 < x \leqslant 1 \end{cases} \tag{2}$$

with the conditions $0 < a < 1$ and $c > 1$, which ensure the existence and give the stability of the three fixed points (the constants are also fixed so that g is continuous):

$$\frac{\alpha}{1-a} < \frac{\gamma}{1-c} < \frac{\beta}{1-a}$$

The results on the existence and the description of the kink solutions for the map g are similar to those obtained below for the map f and we give the final results below. However, the failure of continuity of f may prevent the extension of some results presented here, such as some of the stated properties of the trajectories in the phase space.

In the following, the parameters a, α, β and c in (1) are fixed and the study consists in varying the diffusive coefficient ε in order to describe the symmetry breaking in the set of kink solutions, that is, the front bifurcation that generally develops in this particular bistable dynamical system.

The local map f is a nondifferentiable mapping; more precisely, since the loss of differentiability occurs at c, this point plays a central role in this transition. The CML mapping is then nondifferentiable (when the state vector has a component equal to c) and it is not possible to apply the bifurcation theorems in this case. However, we are able to construct the kink solution using the method of transfer matrices and to sketch the mechanism for the bifurcation that leads to the propagating front structures.

2. THE KINK SOLUTIONS

First, we define a kink to be an orbit $\{x^t\}_{t\in\mathbb{N}}$ of the CML with the properties

$$x_i^t \leqslant x_{i+1}^t \qquad \forall i$$

and

$$\lim_{i\to-\infty} x_i^t = X^1, \qquad \lim_{i\to+\infty} x_i^t = X^2$$

The set of kink K is an invariant set: $F_\varepsilon(K)\subset K$. This comes from the fact that the local map (1) is an increasing function on $[0,1]$ and that X^1 and X^2 are fixed points.

We now consider the set $S(\varepsilon)$ of the steady kink solutions:

$$S(\varepsilon) = S'(\varepsilon) \cup \{x^-, x^+\}$$

where

$$S'(\varepsilon) = \{x^t \in K \mid x^t = x \;\forall t\}$$

and x^- (resp. x^+) is the homogenous solution defined by

$$x_i^- = X^2 \quad \forall i \qquad (\text{resp. } x_i^+ = X^1 \quad \forall i)$$

Notice that, by definition, the kink solutions $x\in S(\varepsilon)$ obey the fixed-point equation $F_\varepsilon x = x$.

Of course, the present study is also valid for the antikink orbits, for which one has to consider instead of K the set

$$AK = \{x' \in M \mid x'_i > x'_{i+1}\ \forall i,\ \lim_{i \to -\infty} x_i = X^2,\ \lim_{i \to +\infty} x_i = X^1\}$$

$S'(\varepsilon)$ is decomposed into the following disjoint subsets, which we consider separately:

$$S'(\varepsilon) = S'_s(\varepsilon) \cup S'_u(\varepsilon)$$

where

$$S'_s(\varepsilon) = \{x \in S'(\varepsilon) \mid \forall i\ x_i \neq c\}$$

and

$$S'_u(\varepsilon) = \{x \in S'(\varepsilon) \mid \exists j\ x_j = c\}$$

Let T be the space translation operator:

$$T\colon\quad M \to M$$
$$x \mapsto Tx$$

where $(Tx)_i = x_{i+1}\ \forall i$.

One can check that F_ε and T commute. Consequently, the subsets $S'_s(\varepsilon)$ and $S'_u(\varepsilon)$ are (globally) invariant under the action of T. We shall see that each of these subsets is entirely determined by any given element, i.e., each is the orbit under T of a single fixed point. If one also notes that the homogeneous fixed points are (pointwise) invariant under the space translations, one can deduce the (global) invariance of $S(\varepsilon)$ under the actions of T.

Let $j \in \mathbb{Z}$. For the sake of simplicity, we denote by $x^j_s \in S'_s(\varepsilon)$ and $x^j_u \in S'_u(\varepsilon)$ the particular kink solutions with the properties

$$\begin{cases} x^j_{s,i} < c & \forall i < j \\ x^j_{s,i} > c & \forall i \geq j \end{cases} \tag{3a}$$

and

$$\begin{cases} x^j_{u,i} < c & \forall i < j \\ x^j_{u,j} = c & \\ x^j_{u,i} > c & \forall i > j \end{cases} \tag{3b}$$

According to these definitions the set $S'(\varepsilon)$ is explicitly given by

$$S'(\varepsilon)=\bigcup_{j\in\mathbb{Z}}\{x_s^j, x_u^j\}$$

since we shall prove that, for each j, there are unique x_s^j and $x_u^j S'(\varepsilon)$.

We now describe the computation of x_s^j using a two-dimensional area-preserving map. First, we introduce the deviation $y=(y_i)_{i\in\mathbb{Z}}$ from the (local map) fixed points:

$$y_i=\begin{cases} x_{s,i}^j-\dfrac{\alpha}{1-a} & \forall i<j \\ \dfrac{\beta}{1-a}-x_{s,i}^j & \forall i\geqslant j \end{cases}$$

The computation of x_s^j components then reduces to the problem of determining the sequences of vectors of the plane that are related by the linear transformations

$$\begin{pmatrix} y_{i-1} \\ y_i \end{pmatrix}=A_\varepsilon\begin{pmatrix} y_i \\ y_{i+1} \end{pmatrix} \qquad \forall i<j-1 \tag{4a}$$

and

$$\begin{pmatrix} y_{i+1} \\ y_i \end{pmatrix}=A_\varepsilon\begin{pmatrix} y_i \\ y_{i-1} \end{pmatrix} \qquad \forall i>j \tag{4b}$$

where

$$A_\varepsilon=\begin{pmatrix} (2/a\varepsilon)(1-a+a\varepsilon) & -1 \\ 1 & 0 \end{pmatrix}$$

is a 2×2 hyperbolic matrix.

According to the boundary conditions for the elements of $S_s'(\varepsilon)$, the y components must vanish at both $+\infty$ and $-\infty$. This implies that the vectors in the relations (4) must be in the contracting (eigen-) direction of A_ε. Therefore the components of y are

$$y_i=\begin{cases} y_{j-1}(\lambda_-)^{j-i-1} & \forall i<j \\ y_j(\lambda_-)^{i-j} & \forall i\geqslant j \end{cases}$$

where λ_- is the eigenvalue of A_ε that is less than one:

$$\lambda_-=\frac{1-a+a\varepsilon-[(1-a)(1-a+2a\varepsilon)]^{1/2}}{a\varepsilon}$$

The computation of the constants y_{j-1} and y_j is performed by writing the affine transformations which corresponds to the connection between the sites above c and those below c:

$$\begin{pmatrix} y_j \\ y_{j-1} \end{pmatrix} = \begin{pmatrix} -\dfrac{2}{a\varepsilon}(1-a+a\varepsilon) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_{j-1} \\ y_{j-2} \end{pmatrix} + \begin{pmatrix} \dfrac{\beta-\alpha}{a(1-a)} \\ 0 \end{pmatrix} \tag{5a}$$

$$\begin{pmatrix} y_{j+1} \\ y_j \end{pmatrix} = \begin{pmatrix} \dfrac{2}{a\varepsilon}(1-a+a\varepsilon) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_j \\ y_{j-1} \end{pmatrix} - \begin{pmatrix} \dfrac{\beta-\alpha}{a(1-a)} \\ 0 \end{pmatrix} \tag{5b}$$

The composition of the relation (5) leads to a new affine transformation between the initial vectors of the maps (4). This new map formally reads

$$\begin{pmatrix} y_{j+1} \\ y_j \end{pmatrix} = \mathscr{A} \begin{pmatrix} y_{j-1} \\ y_{j-2} \end{pmatrix} + T$$

The matrix A and the vector T are obviously deduced from those of relation (5). In order to ensure the existence and consequently the uniqueness of the solution, we check, using the properties

$$\begin{pmatrix} y_{j+1} \\ y_j \end{pmatrix} = y_j \begin{pmatrix} \lambda_- \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y_{j-1} \\ y_{j-2} \end{pmatrix} = y_{j-1} \begin{pmatrix} 1 \\ \lambda_- \end{pmatrix}$$

that the vectors

$$\begin{pmatrix} y_{j+1} \\ y_j \end{pmatrix} \quad \text{and} \quad \mathscr{A} \begin{pmatrix} y_{j-1} \\ y_{j-2} \end{pmatrix}$$

are linearly independent. This calculation leads to the simple condition $a < 1$. Therefore the existence and uniqueness of the kink solution is given by the stability of the local map fixed points. The computation effectively reduces to the resolution of a system of two linear equations in y_{j-1} and y_j and the components of x_s^j finally read

$$x_{s,i}^j = \begin{cases} \dfrac{\alpha}{1-a} + \dfrac{\beta-\alpha}{a(1-a)(1+\lambda_-)} (\lambda_-)^{j-i} & \forall i < j \\ \dfrac{\beta}{1-a} - \dfrac{\beta-\alpha}{a(1-a)(1+\lambda_-)} (\lambda_-)^{i-j+1} & \forall i \geqslant j \end{cases} \tag{6}$$

However, some restrictions on this solution may be imposed by the conditions (3a). Actually it will be shown below that these limitations are crucial, as they give the bifurcation point since (3a) and (6) are compatible only if ε is smaller than a critical value defined in Section 4.

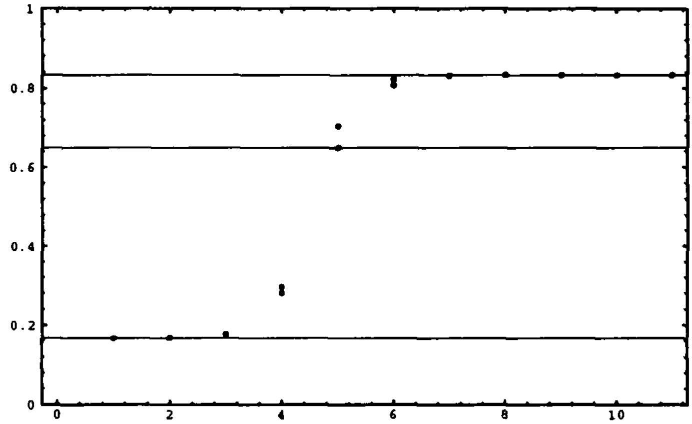

Fig. 2. The stable x_s^j and the unstable x_u^j fixed points for $\varepsilon = 0.3$. The parameters for f are $a = 0.4$, $\alpha = 0.1$, $\beta = 0.5$, and $c = 0.65$. The horizontal lines stand, respectively, for X^1, c, and X^2.

The same method is applied in order to compute the components of x_u^j. Also in this case the determination of the constants y_{j-1} and y_{j+1} gives an equation similar to the system (5). Thanks to the particular definition of $f(c)$, the condition for the existence and uniqueness of the solution is also $a < 1$ and x_u^j reads

$$x_{u,i}^j = \begin{cases} \dfrac{\alpha}{1-a} + \dfrac{1}{a}\left[f(c) - \dfrac{\alpha}{1-a} \right] (\lambda_-)^{j-i} & \forall i < j \\ c & i = j \\ \dfrac{\beta}{1-a} - \dfrac{1}{a}\left[\dfrac{\beta}{1-a} - f(c) \right] (\lambda_-)^{i-j} & \forall i > j \end{cases} \tag{7}$$

A plot of these solutions is given in Fig. 2. Notice that the uniqueness also implies that the system (3b) has a unique solution for $\varepsilon < \varepsilon_c$ where the critical value ε_c will be specified in Section 4.

For the corresponding antikink structures, the elements of $S''(\varepsilon)$ can be deduced from those of $S'(\varepsilon)$ by simply applying the symmetries R_s^j for the points x_s^j and R_u^j for x_u^j, where

$$(R_s^j x)_{j+i} = x_{j-i-1} \qquad \forall i$$

and

$$(R_u^j x)_{j+i} = x_{j-i} \qquad \forall i$$

3. THE STABILITY ANALYSIS

The stability of the fixed points x_s^j and x_u^j is now investigated. We show that x_s^j is stable whereas x_u^j is unstable; more precisely, the former is a saddle. Here again due to the translational invariance along the lattice the following results may, with some caution, be extended to any other element of $S_s^j(\varepsilon)$ and $S_u^j(\varepsilon)$. The study is performed by the computation of the perturbation dynamics in a neighborhood of these solutions. This approach naturally leads to the description of the stable manifold of x_s^j and x_u^j. Noticing that the central manifold is empty the unstable manifold may be deduced from the stable one.

The sets of perturbations under consideration are

$$V_s^j = \{P \in \mathbb{R}^{\mathbb{Z}} \mid x = x_s^j + P, x \in M, x_i < c\ \forall i < j \text{ and } x_i > c\ \forall i \geqslant j\}$$

in the stability analysis of x_s^j and

$$V_u^j = \{P \in \mathbb{R}^{\mathbb{Z}} \mid x = x_u^j + P, x \in M, x_i < c\ \forall i < j, x_j \in [0, 1] \text{ and } x_i > c\ \forall i > j\}$$

for the point x_u^j. These sets overlap from one fixed point to the other.

We adopt the usual definition of the local stable manifold, but we may express it in terms of perturbations:

$$W_{\text{loc}}^s(x_*^j) = x_*^j + \{P \in V_*^j \mid \mathscr{F}_\varepsilon^t(P) \to 0 \text{ as } t \to +\infty \text{ and } \mathscr{F}_\varepsilon^t(P) \in V_*^j\ \forall t \geqslant 0\}$$

where $*$ stands either for s or u, and the perturbation map is defined by

$$\mathscr{F}_\varepsilon : V_*^j \to \mathbb{R}^{\mathbb{Z}}$$
$$P \mapsto \mathscr{F}_\varepsilon(P) = F_\varepsilon(x_*^j + P) - x_*^j$$

Moreover, the stable manifold is

$$W^s(x_*^j) = \bigcup_{t \geqslant 0} \mathscr{F}_\varepsilon^{-t}(W_{\text{loc}}^s(x_*^j))$$

This definition is independent of the original neighborhood. The computation of these sets is cumbersome and relatively useless. Indeed, the invariant manifolds in the case of maps are generically sets of isolated points, and thus are not manifolds in the usual sense. We avoid the problem of describing all the trajectories by restricting the initial conditions for which the orbits stay in the neighborhoods $x_s^j + V_s^j$ and $x_u^j + V_u^j$.

In the case of the point x_s^j we obtain $\mathscr{F}_\varepsilon(P) = J_\varepsilon P$, where J_ε is the tridiagonal (infinite-dimensional) operator:

$$J_\varepsilon = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & 0 & \varepsilon a/2 & (1-\varepsilon)\,a & \varepsilon a/2 & 0 & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

For x_u^j the dynamics is more subtle, as it contains the local map discontinuity:

$$\forall P \in V_u^j \qquad \mathscr{F}_\varepsilon(P) = \begin{cases} J_\varepsilon P + \eta_- & \text{if } \; P_j < 0 \\ J_\varepsilon P & \text{if } \; P_j = 0 \\ J_\varepsilon P + \eta_+ & \text{if } \; P_j > 0 \end{cases}$$

where η_- and η_+ are the vectors of $\mathbb{R}^{\mathbb{Z}}$ whose only nonvanishing components are the following

$$\eta_{-,i} = \begin{cases} \frac{1}{2}\varepsilon[ac + \alpha - f(c)] & i = j \pm 1 \\ (1-\varepsilon)[ac + \alpha - f(c)] & i = j \end{cases}$$

and

$$\eta_{+,i} = \begin{cases} \frac{1}{2}\varepsilon[ac + \beta - f(c)] & i = j \pm 1 \\ (1-\varepsilon)[ac + \beta - f(c)] & i = j \end{cases}$$

The stability of x_s^j is given by the following result.

Proposition 3.1. $W_{\text{loc}}^s(x_s^j) = x_s^j + V_s^j$.

This assertion implies that $W_{\text{loc}}^u(x_s^j)$ is empty, hence n_s^j is stable, i.e., it is a node. This is the observed solution in numerical simulations.

Proof. Here we use the notation V_s^j for $x_s^j + V_s^j$.

By construction, one has $W_{\text{loc}}^s(x_s^j) \subset V_s^j$.

We endow $\mathbb{R}^{\mathbb{Z}}$ with the inner product[12]

$$\langle x, y \rangle_q = \sum_{i \in \mathbb{Z}} \frac{x_i y_i}{q^{|i|}} \qquad \text{for any} \quad q > 1$$

and the norm $\|\cdot\|_q = (\langle \cdot, \cdot \rangle_q)^{1/2}$. We consider the Hilbert space $B_q = \{P \in \mathbb{R}^{\mathbb{Z}} \mid \|P\|_q < \infty\}$ and $\|\cdot\|$ the usual supremum norm for operators. We have

$$\|\mathscr{F}_\varepsilon(P)\|_q \leqslant \|J_\varepsilon\| \cdot \|P\|_q, \qquad \forall P \in B_q$$

J_ε is a normal operator, hence we get

$$\|J_\varepsilon\| = r(J_\varepsilon) = \sup_{\lambda \in \sigma(J_\varepsilon)} |\lambda|$$

where $\sigma(J_\varepsilon)$ is the spectrum of J_ε. Using the method developed in ref. 12 one can deduce that

$$\sigma(J_\varepsilon) \subset \text{Clos}\left\{\bigcup_{N \geqslant N_0} \Lambda_N\right\} \cup \{0\}$$

where

$$\Lambda_N = \left\{a(1-\varepsilon) + a\varepsilon \cos \frac{k\pi}{N+1}, k = 1, N\right\}$$

is the spectrum of the finite-dimensional approximation of J_ε, that is, the spectrum of the tridiagonal matrix of size N.

Therefore $r(J_\varepsilon) = |a| = a < 1$. This gives the required statement $\mathscr{F}_\varepsilon(P) \in V_s^j \; \forall P \in V_s^j$. The first condition for an element of V_s^j to belong to $W_{\text{loc}}^s(x_s^j)$ has been checked. The second one is also valid when writing

$$\mathscr{F}_\varepsilon^t(P) = (J_\varepsilon)^t \, P$$

from which it clearly follows that $\mathscr{F}_\varepsilon^t(P) \to 0$ as $t \to +\infty$. Both of these assertions imply $V_s^j \subset W_{\text{loc}}^s(x_s^j)$, which ends the proof. ∎

We consider the following useful properties for the decomposition of V^j_*:

Definition 3.2. $P \in V^j_*$ is symmetric (resp. skew-symmetric) if $P_{j+i} = P_{j-i} \; \forall i$ (resp. $P_{j+i} = -P_{j-i} \; \forall i$). The symmetric (resp. skew-symmetric) vectors are denoted P_s (resp. P_a).

This definition is motivated by the conservation of some symmetries under the action of J_ε; clearly $J_\varepsilon \cdot P_s$ is symmetric and $J_\varepsilon \cdot P_a$ is skew-symmetric.

Write $V_a^j = \{P \in V_a^j | P = P_a\}$, the subset of skew-symmetric perturbations. The stability of x_u^j is given by the following:

Proposition 3.3. $W_{\text{loc}}^s(x_u^j) = x_u^j + V_a^j$.

Proof. By induction. One has $\forall P \in V_a^j \; P_j = 0$. Then $\mathscr{F}_\varepsilon(P) = J_\varepsilon P$, which is known to have the required properties of being in the local stable manifold.

Moreover, suppose that $\mathscr{F}_\varepsilon^t(P) \in V_a^j$; then

$$\|\mathscr{F}_\varepsilon^{t+1}(P)\|_q = \|\mathscr{F}_\varepsilon(\mathscr{F}_\varepsilon(P))\|_q \leqslant \|J_\varepsilon\| \cdot \|\mathscr{F}_\varepsilon(P)\|_q$$

from which we deduce $V_a^j \subset W_{\text{loc}}^s(x_u^j)$, where we discard x_u^j in the notation.

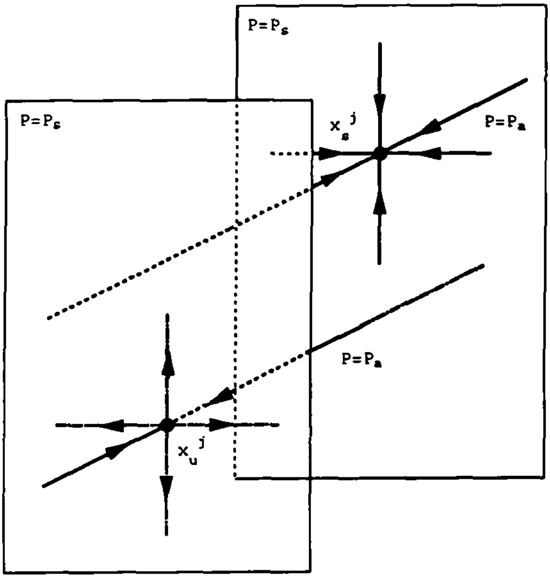

Fig. 3. Schematic three-dimensional representation of a phase space region. The solid lines stand for the stable directions of the (local) stable manifold. The dashed lines represent the unstable local manifold.

The proof that $W^s_{\rm loc}(x^j_u) \subset V^j_a$ is similar to the previous one. We show that in order to be (always) decreasing and asymptotically vanishing, a perturbation must have a vanishing jth component after each iteration. This condition implies the result. ∎

Hence we have shown the point x^j_u to be a saddle. A three-dimensional schematic representation of the phase portrait of these two fixed points is displayed in Fig. 3. The connection to the other points may not be so easy because the symmetric and skew-symmetric axes vary from one stable (or unstable) fixed point to the other.

Finally we have the following:

Proposition 3.4. $\forall P \in V^j_*$ such that $\forall t$, $\mathscr{F}^t_\varepsilon(P) \in V^j_*$, $\lim_{t \to +\infty} \|\mathscr{F}^t_\varepsilon(P)\|_q \leqslant D$ where

$$D = \max\{\|x^j_s - x^j_u\|_q, \|x^{j+1}_s - x^j_u\|_q\}$$

This statement ensures that all suitable orbits evolve toward a point in $S'(\varepsilon)$, and thus cannot reach the homogeneous state. [We have not investigated the cases where the orbit leaves the neighborhood under consideration, but we conjecture that it will be trapped in a neighborhood of another fixed solution in $S'(\varepsilon)$ and stay inside forever.]

Proof. From the proof of Proposition 3.1 the case $P \in V_s^j$ obeys the statement, since for such perturbations, the dynamics is simply given by the product with the contracting matrix J_ε. In this situation, the limit in norm is zero and the asymptotic state is x_s^j. The case $P \in V_u^j$ also follows from Proposition 3.3 by the same argument and the final state is x_u^j.

Assume that $P \in V_u^j$ is such that $P_i > 0$ $\forall i$ (resp. $P_i < 0$ $\forall i$). Then for $\|J_\varepsilon\| < 1$, the asymptotic state for such a perturbation is given by

$$\lim_{t \to +\infty} \mathscr{F}_\varepsilon^t(P) = \sum_{t=0}^{\infty} J_\varepsilon^t \eta_+ = (Id - J_\varepsilon)^{-1} \eta_+ = x_s^j - x_u^j$$

[resp.

$$\lim_{t \to +\infty} \mathscr{F}_\varepsilon^t(P) = (Id - J_\varepsilon)^{-1} \eta_- = x_s^{j+1} - x_u^j]$$

Id means the identity operator in V_u^j. Consequently, the norm limit is one of the values of D depending on the sign of the P components.

Now for any $P \in V_u^j$, one has in a componentwise sense

$$(J_\varepsilon)^t \cdot P + \sum_{k=0}^{t-1} J_\varepsilon^k \eta_- \leqslant \mathscr{F}_\varepsilon^t(P) \leqslant (J_\varepsilon)^t \cdot P + \sum_{k=0}^{t-1} J_\varepsilon^k \eta_+ \qquad \forall t > 0$$

This inequality implies the asymptotic boundedness. ∎

4. THE CONDITIONS FOR THE EXISTENCE OF THE KINKLIKE FIXED POINTS AND THE GENERALIZED SADDLE-NODE BIFURCATION

The kinklike fixed points have been computed assuming the conditions (3). However, these assumptions have to be checked afterward as the expressions (6) and (7) for the components of x_s^j and x_u^j mainly depend on ε. Some properties of the components (6) will allow us to claim a criterion for the existence of the fixed interfaces in our CML. One can check that de expressions (6) obey the following.

Proposition 4.1. $\forall i < j$ (resp. $\forall i \geqslant j$) the components of x_s^j are increasing (resp. decreasing) functions of the coupling strength ε.

Let

$$K_\alpha \equiv \frac{c(1-a) - \alpha}{\beta - \alpha}, \qquad K_\beta \equiv \frac{\beta - c(1-a)}{\beta - \alpha}, \qquad \varepsilon_{\alpha,\beta} \equiv \frac{2(1-a)\, K_{\alpha,\beta}(1 - aK_{\alpha,\beta})}{(1 - 2aK_{\alpha,\beta})^2}$$

The following are true:

$$\forall \varepsilon > \varepsilon_\alpha \qquad x^j_{s,j-1} > c$$

$$\forall \varepsilon > \varepsilon_\beta \qquad x^j_{s,j} < c$$

The proof is accomplished with simple calculations.

This proposition implies that the fixed points x^j_s no longer exist for $\varepsilon > \varepsilon_c \equiv \min\{\varepsilon_\alpha, \varepsilon_\beta\}$. Moreover, the image of c has been constructed in such a way that the points x^j_u also no longer exist when $\varepsilon > \varepsilon_c$. This is because the problem of the transfer matrices for the saddle points has no solution for this range of diffusive coefficient as $f(c) = \alpha c + \beta$.

In other words, we have described a (multi) generalized saddle-node bifurcation that occurs for all the kinklike fixed points in our bistable CML. This bifurcation can be viewed as a transition from a global translational symmetry invariance in the set of fixed interfaces

$$S(\varepsilon) = \bigcup_{j \in \mathbb{Z}} \{x^j_s, x^j_u\} \cup \{x^-, x^+\} \qquad \forall 0 \leqslant \varepsilon \leqslant \varepsilon_c$$

to a pointwise translational symmetry of

$$S(\varepsilon) = \{x^-, x^+\} \qquad \forall \varepsilon_c < \varepsilon \leqslant 1$$

The resulting attractors for a kinklike initial condition may be one of the homogeneous solutions when $\varepsilon > \varepsilon_c$. Indeed, the analysis of the perturbation dynamics near a fixed point $x^j \equiv x^j_s = x^j_u$ at $\varepsilon = \varepsilon_c$ may give an insight into this property.

Let $c > (X^1 + X^2)/2$; then $\varepsilon_c = \varepsilon_\beta$. Note that the case of equality is the symmetric case where the fixed fronts always exist (that is, for any $\varepsilon \in [0,1]$, and that the case $c < (X^1 + X^2)/2$ is achieved in the same way. The dynamics for a perturbation of the fixed point x^j reads

$$\mathscr{F}_{\varepsilon_c}(P) = \begin{cases} J_{\varepsilon_c} P & \text{if } \ P_j \geqslant 0 \\ J_{\varepsilon_c} P + \nu^j_- & \text{if } \ P_j < 0 \end{cases}$$

where ν^j_- is the vector η_- computed at ε_c:

$$(\nu^j_-)_i = \begin{cases} 0, & i < j-1 \text{ or } i > j+1 \\ \frac{1}{2}\varepsilon_c(\alpha - \beta), & i = j \pm 1 \\ (1 - \varepsilon_c)(\alpha - \beta), & i = j \end{cases}$$

Every perturbation with positive components is damped and the asymptotic state is x^j. However, any perturbation with negative components is not damped and as time evolves it approaches the value [see the proof (3.4)]

$$\lim_{t \to +\infty} \mathscr{F}^t_{\varepsilon_c}(P) = x^{j+1} - x^j$$

The asymptotic state of the system is x^{j+1} in this case. Therefore any kinklike initial condition, that is, any initial condition in the basin of attraction of one of the x^j, evolves toward the "right" (from the lattice point of view) and reaches one of the fixed points asymptotically. By contrast, any antikinklike initial condition may propagate to the "left", as can be seen from a similar perturbation analysis of points $S''(\varepsilon)$ at $\varepsilon = \varepsilon_c$. Hence, according to the sign of the quantity $c - (X^1 + X^2)/2$, one might decide on the direction of the front and the antifront propagation for the coupling above the critical value.

5. THE TRANSITION FOR CONTINUOUS LOCAL MAPS

In this section, we describe the steady-propagating front transition for continuous local maps. The first situation deals with the map g [defined in (2)]; then we consider numerically the case of a differentiable mapping.

If the local map is chosen to be the map g, the same analysis as above can be done, that is, the calculation of the points x^j_s and x^j_u, the analysis of stability, and the bifurcation. In this case, the x^j_s components are also given by the system (6), whereas we only consider the unstable solution with one component in the interval $]c_1, c_2[$. One can check that x^j_u exists and is unique. The condition for the existence of the steady front is also $\varepsilon < \varepsilon_c \equiv \min\{\varepsilon_\alpha, \varepsilon_\beta\}$, where ε_α and ε_β are defined as in Proposition 4.1 but with different values of K_α and K_β:

$$K_\alpha = \frac{c_1(1-a) - \alpha}{\beta - \alpha} \qquad \text{and} \qquad K_\beta = \frac{\beta - c_2(1-a)}{\beta - \alpha}$$

The stability analysis of x^j_s also implies that it is a stable point. The investigation of the perturbation dynamics is not so simple for x^j_u but again it is possible to show that it is a saddle. This result is confirmed by the numerical computation of the associated linear dynamics spectrum. Hence, the CML dynamics also reveals a generalized saddle-node bifurcation in this case.

One step further in the complexity of the local dynamics is to examine a differentiable bistable map, a model that is closer to a more realistic

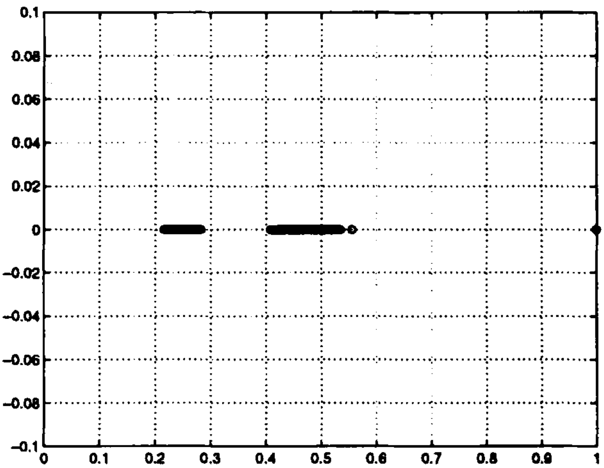

Fig. 4. Spectrum of the kinklike fixed point Jacobian (imaginary part vs. real part). The parameters are $\mu = 1.3$, $c = 0.02$, and $\varepsilon = 0.1193$.

situation. Here we have chosen the (nonsymmetric) cubic map $h_{(\mu, c)}(x) = c + \mu x(1 - x^2)$. For suitable values of μ and c, $h_{(\mu, c)}$ is also bistable. In this context, we have no idea how to compute explicitly the components of the fixed point. Indeed, the method of transfer matrices is no longer appropriate because the relation between the neighbors is quadratic.

However, due to the bistable feature, the CML with the map $h_{(\mu, c)}$ may reveal the same bifurcation as in the former cases. In order to check this claim, we have computed numerically the spectrum of the Jacobian associated with the kink fixed point. The result is presented in Fig. 4, where it clearly appears that the greatest Jacobian eigenvalue occurs at one for $\varepsilon = 0.1193$, the mark of a saddle-node bifurcation in differentiable cases. This value of the coupling exactly corresponds to the value at which the front propagates in the lattice, as can be seen from the simulations. Notice the interesting and somewhat unexpected result (see Fig. 4) that the spectrum shows an isolated eigenvalue that crosses the unit circle and is isolated from the remainder of the spectrum by a uniform gap.

6. THE OTHER FIXED POINTS

In this section, we give an insight into the other types of steady solutions inherent to the CML under consideration, that is, with the local map (1). The fixed-point equation is expressed as

$$G(\varepsilon, x) = 0 \tag{8}$$

with $G(\varepsilon, x) = F_\varepsilon x - x$, and we denote by $\mathscr{S}(\varepsilon)$ the set of solutions depending on the coupling strength. We endow the extended phase space $\mathbb{R}^{\mathbb{Z}}$ with the usual norm:

$$\|x\|_\infty = \sup_{i \in \mathbb{Z}} |x_i|$$

and from now on we consider the Banach space $B_\infty = \{x \in \mathbb{R}^{\mathbb{Z}} \mid \|x\|_\infty < \infty\}$. For $\varepsilon = 0$ the dynamics consist of a set of uncoupled maps $(F_0 x')_i = f(x'_i)$; this yields

$$\mathscr{S}(0) = \{x \in [0, 1]^{\mathbb{Z}} \mid \forall i\; x_i = X^1 \text{ or } X^2\}$$

which means that the system has the property of spatial chaos.[13]

The continuation of the fixed points into the coupled case is guaranteed by the application of the Implicit Function Theorem to Eq. (8) at each point x_0 of $\mathscr{S}(0)$.[14, 15] We describe now the conditions for the use of this theorem and its consequence.

Let $U(0, x_0) \in (\mathbb{R}, |\cdot|) \times B_\infty$ be an open neighborhood of $(0, x_0)$ such that:

(i) The (infinite) Jacobian $DG(0, x_0)$ exists as a Frechet derivative on $U(0, x_0)$ and is invertible.

(ii) G and DG are continuous at $(0, x_0)$.

The conclusion is then that there exists a number δ such that for every ε satisfying $|\varepsilon| < \delta$ there is exactly one $x(\varepsilon)$ for which $G(\varepsilon, x(\varepsilon)) = 0$. Note that, thanks to the linearity of f, the theorem also gives an exact bound on $x(\varepsilon)$. Furthermore, as $G(\varepsilon, x)$ is continuous in a neighborhood of $(0, x_0)$, $x(\varepsilon)$ is continuous in a neighborhood of 0.

Here we choose

$$U(0, x_0) =]0, \delta[\times \prod_{i \in \mathbb{Z}} I_i$$

where

$$I_i = \begin{cases}]0, c[& \text{if } (x_0)_i = X^1 \\ ]c, 1[& \text{if } (x_0)_i = X^2 \end{cases}$$

The main condition for the continuation of x_0 is (i); thus the fixed points may exist as long as F_ε is differentiable. This condition fails when (at least) one component of $x(\varepsilon)$ is c. Hence, we may obtain a condition for the existence of any fixed point similar to the one computed for the kink solution (Proposition 4.1). The bound δ depends on the particular point x_0, but it is possible to obtain a uniform bound for any solution.[14] Moreover, not only is $x(\varepsilon)$ unique, but since F_ε is contracting on $U(0, x_0)$, there is no other possible fixed solution in this set. Note also that the Implicit Function Theorem can also be applied in the case of a differentiable local map. In such a case, $G(\varepsilon, x)$ is always differentiable and the fixed points exist as long as $DG(\varepsilon, x(\varepsilon))$ is invertible, that is, until the spectrum of DF_ε lies entirely within the unit circle.

Finally, as in the case of the front, the critical values of ε (for which the solution disappears) are given for two examples of the CML defined with the map (1). This is done by generalizing the transfer matrix technique and by checking afterward the conditions for the existence of the solution. Here we suppose again for the sake of definiteness that $c > (X^1 + X^2)/2$ (the opposite case can be handled in a similar manner). For the one-point domain solution which is defined by

$$\exists j \quad \text{such that } x_j > c \text{ and } \forall i \neq j \quad x_i < c$$

the critical value is

$$\varepsilon'_\beta = \frac{(1-a)\, K_\beta(2 - aK_\beta)}{2(1 - aK_\beta)^2}$$

where K_β is given in (4.1). This solution is (numerically) the less stable fixed point in the structural sense, that is, the first solution to disappear when one increases ε from 0. For the (spatial) 2-periodic point

$$\forall i \qquad x_{2i} > c \quad \text{and} \quad x_{2i+1} < c$$

we have found

$$\varepsilon''_\beta = \frac{(1-a)\, K_\beta}{1 - 2aK_\beta}$$

and we obtain the following ordering of the critical values:

$$\varepsilon'_\beta < \varepsilon''_\beta < \varepsilon_\beta$$

from which we conjuncture that the kink solution has the largest transition value.

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REFERENCES

1. J. D. Keeler and J. D. Farmer, Robust space-time intermittency and $1/f$ noise, *Physica D* **23**:413–435 (1986).
2. A. Lambert and R. Lima, Stability of wavelengths and spacio-temporal intermittency, *Physica D* **71**:390–411 (1994).
3. M. C. Cross and P. C. Hohenberg, Pattern formation outside equilibrium, *Rev. Mod. Phys.* **65**:851–1113 (1993).
4. P. Collet and J. P. Eckmann, *Instabilities and Fronts in Extended Systems* (Princeton University Press, Princeton, New Jersey, 1990).
5. W. V. Saarloos, Front propagation into unstable states: Marginal stability as a dynamical mechanism for velocity selection, *Phys. Rev. A* **37**:211–229 (1988).
6. B. Denardo, B. Galvin, A. Greenfield, A. Larraza, S. Putterman, and W. Wright, Observation of localized structures in nonlinear lattices: Domains walls and kinks, *Phys. Rev. Lett.* **68**:1730–1733 (1992).
7. Y. S. Kivshar, Class of localized structures in nonlinear lattices, *Phys. Rev. B* **46**:8652–8654 (1992).
8. J. P. Crutchfield and K. Kaneko, Phenomenology of spatio-temporal chaos, *Direction in Chaos* Hao Bai-Lin, ed. (World Scientific, Singapore, 1987), pp 272–353.
9. K. Kaneko, The coupled map lattices, in *Theory and Applications of Coupled Map Lattices* K. Kaneko, ed. (Wiley, New York, 1993).
10. B. Fernandez, Kink dynamics in one-dimensional coupled map lattice, *Chaos* **5**(3):602–608 (1995).
11. S. Aubry, D. Escande, J. P. Gaspard, P. Manneville, and J. Villain, *Structures et Instabilités* (Les éditions de physique, Orsay, 1986).
12. V. S. Afraimovich and L. A. Bunimovich, Simplest structures in coupled map lattices and their stability, *Random Computational Dynamics* **1**:423–444 (1993).
13. V. S. Afraimovich, L. Y. Glebsky, and V. I. Nekorkin, Stability of stationary states and topological spatial chaos in multidimensional lattice dynamical systems, *Random Computational Dynamics* **2**:287–303 (1994).
14. R. S. Mackay and J.-A. Sepulchre, Multistability in networks of weakly coupled bistable units, *Physica D* **82**:243–254 (1995).
15. V. S. Afraimovich and L. A. Bunimovich, Density of defects and spatial entropy in extended systems, *Physica D* **80**:277–288 (1995).